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A NUMERICAL EVALUATION OF THE LIOUVILLE-GREEN APPROXIMATION OF VARIABLE-COEFFICIENT LANCHESTER-TYPE EQUATIONS OF MODERN WARFARE

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THESIS

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by

James N. Carpenter

March, 1976

Thesis Advisor: J.G. Taylor

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of the two opposing weapon systems. These coefficients might be used to model a constant-speed attack against a static defensive position. It is shown that for these attrition-rate coefficients, the Liouville-Green approximation is not consistently reliable for predicting force levels, and yields exact results only under certain restrictive conditions. Furthermore it was found that methodology is not presently available to accurately predict from Liouville's normal form the error which will be incurred by invoking the approximation in a specific situation.

A Numerical Evaluation of the Liouville-Green Approximation of Variable-Coefficient Lanchester-Type Equations of Modern Warfare

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I. INTRODUCTION

Although it is well recognized that combat is a complex random process, a model that provides "closed-form" solutions can often be an aid to the military analyst by explictly portraying interrelationships between combat parameters. Deterministic differential equations are frequently used to model the combat process, and vary greatly in their detail and level of complexity. The military analyst must be able to recognize significant variables and their relationships. An idealized model is useful for giving a "first cut" portrayal of such relationships. In particular, one can often obtain an analytical sclution to such models. An analytical solution that can be expressed in terms of "elementary" functions can often provide insights into the combat process and identify factors affecting battle outcome.

The use of differential equations to model warfare was pioneered by F. W. Lanchester in 1914 [Ref 8]. This original work was quite idealized and assumed that the fire effectivenesses were constant throughout the battle. Many extensions to Lanchester's classic formulation have subsequently been developed, to include such factors as mobility of forces and fire effectiveness that varies during the battle. Taylor and Brown [Ref 13] have developed a mathematical theory for solving Lanchester-type equations with these temporal variations in fire effectiveness. The fire effectiveness of forces is represented by what are now called Lanchester attrition-rate coefficients. Taylor and Brown give results for two specific forms of the coefficients: (1) effectiveness of each side's fire proportional to a power of time, and (2) effectiveness of each side's fire linear with time but a nonconstant ratio of attrition- rate coefficients.

Taylor and Brown present accurate numerical solutions to equations with the above attrition-rate coefficients. These solutions are, in general, stated in terms of infinite series. The solution method, however, is rather complicated and involved the development of new mathematical functions alien to the military analyst. Exact results are certainly beneficial, but the analyst also needs a solution in terms of "familiar" functions to permit parametric analysis and simplify recognition of significant variables. To provide such a solution form, Taylor [Ref 14] has suggested the use of the so-called Liouville-Green approximation, which can be expressed in terms of relatively simple functions. For an approximation to be useful in predicting force levels, however, it must be reasonably "close" to the exact solution.

This paper provides a numerical evaluation of the Liouville-green approximation for the two forms of attrition-rate coefficients mentioned above. A comparison is made between the approximation in each of these cases and the numerical results given by Taylor and Brown. Numerical examples and analytic considerations will be utilized to investigate the accuracy of the approximation in these cases.

The organization of this thesis is as follows. We first review Lanchester's classic model of combat between two homogeneous forces. These results are then extended to the variable attrition-rate coefficient case, and the specific forms to be studied are discussed. The representation of the solution to these variable-coefficient equations in terms of so-called general Lanchester functions is presented. Then Liouville's normal form and approximation are introduced, evaluated, and discussed for the two specific forms of attrition-rate coefficients.

II. LANCHESTER'S CLASSIC FORMULATION

F. W. Lanchester was an English aeronautical engineer who first developed a mathematical formulation for combat between two conflicting forces. He published this theory in a series of articles in the British journal Engineering in 1914 [Ref 8]. Lanchester's objectives were to provide insights into "modern" combat and to quantitatively justify the military principle of concentration. He stressed that advances in technology had brought long-range delivery capability of modern weapons which allowed for concentration of firepower. Lanchester hypothesized that under these modern conditions, combat between two military forces could be modelled by:

$$dx/dt = -ay \qquad \text{with} \qquad x(t=0) = x_0 ,$$

$$dy/dt = -bx \qquad \text{with} \qquad y(t=0) = y_0 , \qquad (1)$$

where $x(t)$ and $y(t)$ are the numbers of X and Y forces at time t, and $t=0$ denotes the time at which the battle begins. These equations are valid only for x and y greater than zero. The parameters a and b are non-negative constants that have come to be referred to as Lanchester attrition-rate coefficients. These coefficients represent each side's fire effectiveness or firepower. The most common set of circumstances under which these equations have been hypothesized to apply is that both sides use aimed fire and target acquisition times are constant [Ref 16].

From (1) Lanchester deduced his classic square law

$$b(x_0^2 - x^2(t)) = a(y_0^2 - y^2(t)), \qquad (2)$$

which has the important implication that a force can significantly reduce its casualties by initially committing

more forces to the battle. The square law also shows that in this simple model the Y force wins (ie., X is annihilated) if and only if

$$x_0/y_0 < \sqrt{a/b}\ . \tag{3}$$

From (1) Lanchester also derived the now well-known results for the time history of the force levels

$$x(t) = x_0 \cosh\sqrt{ab}\ t - y_0\sqrt{a/b}\ \sinh\sqrt{ab}\ t\ ,$$

$$y(t) = y_0 \cosh\sqrt{ab}\ t - x_0\sqrt{b/a}\ \sinh\sqrt{ab}\ t\ , \tag{4}$$

The victory-prediction condition (3) was given for a "fight to the finish". As H. K. Weiss [Ref 15] points out, most battles terminate before complete annihilation of one force occurs. Different models of battle termination have been proposed, among them that combat ends when either of two given "breakpoint" force ratios is reached. Taylor has shown that (3) is necessary and sufficient for Y to "win" a fixed-force-ratio-breakpoint battle [Ref 14].

Lanchester's original formulation (1) involved many implicit assumptions that limit its usefulness for modelling real combat. The forces were assumed to be homogeneous, there were no replacements or withdrawals, and no movement of forces was considered. Setting the attrition-rate coefficients a and b in (1) as constants implies that the fire effectiveness per firer of each side is constant throughout the battle.

III. VARIABLE ATTRITION-RATE COEFFICIENTS

When modelling mobile weapon systems and movement of forces, one recognizes that the assumption of constant fire effectiveness is open to question [Ref 2]. If fire effectiveness is permitted to vary during the battle, Lanchester's formulation (1) takes the form:

$$dx/dt = -a(t)\ y \qquad \text{with } x(t=0) = x_0\ ,$$
$$dy/dt = -b(t)\ x \qquad \text{with } y(t=0) = y_0\ , \tag{5}$$

where a(t) and b(t) are time-dependent attrition-rate coefficients. The form of the attrition-rate coefficients depends on such variables as force separation (range of battle), tactical posture of the targets, firing rate and rate of target acquisition [Ref 5].

New operations research techniques for predicting these attrition-rate coefficients were developed in the 1960's. A significant contribution in this area was the development of a methodology by S. Bonder [Ref 2 and 4] for prediction of coefficients based on weapon system performance data. Another important development was the method of G. Clark for the maximum likelihood estimation of these coefficients based on Monte Carlo simulation output data [Ref 6]. The work of these and others has facilitated the use of models such as (5) in current combat studies.

To effectively model today's highly mobile warfare, the form of the attrition-rate coefficients in (5) must accomodate current tactics and battle conditions. Specifically, the coefficients must permit the fire effectiveness of one or both sides to be non-zero at the start of battle (to model, for example, an ambush). The coefficients must also allow for the opposing weapon systems to have different maximum effective ranges. Since mobile

forces close with each other in combat, their fire effectiveness should be permitted to increase or decrease as a function of range or time. The attrition-rate coefficients examined in this paper have the flexibility to model these various situations. Motivated by previous work by Bonder and Farrell [Ref 5], Taylor and Brown have considered the following attrition-rate coefficients:

$$a(t) = k_a (t+C)^m \quad \text{and} \quad b(t) = k_b (t+C+A)^n. \tag{6}$$

The specific cases of (6) to be considered here will be that in which A=0, C≥0,

$$a(t) = k_a (t+C)^m \quad \text{and} \quad b(t) = k_b (t+C)^n, \tag{7}$$

and that in which m=n=1, A>0, C≥0,

$$a(t) = k_a (t+C) \quad \text{and} \quad b(t) = k_b (t+C+A). \tag{8}$$

The parameter A will be referred to as the "offset parameter" since it permits modelling of combat between forces whose weapon systems have different maximum effective ranges. The parameter A also allows the ratio of attrition-rate coefficients to be non-constant. Koopman [see Ref 10] first observed that if the ratio of attrition-rate coefficients is constant, the time solution to (5) is no more complicated than the constant-coefficient solution. That is, when

$$a(t) = k_a h(t) \quad \text{and} \quad b(t) = k_b h(t), \tag{9}$$

where h(t) is the common time-dependent factor in both attrition-rate coefficients, the solution is

$$x(t) = x_0 \cosh W(t) - y_0 \sqrt{k_a/k_b} \sinh W(t),$$

where $W(t) = \sqrt{k_a k_b} \int_0^t h(s)\,ds$. Koopman also found that the constant ratio of coefficients again yields a square-law relationship(see also Ref 16).

The parameter C in (7) and (8) allows for the initial fire effectiveness of each side to be non-zero. This permits modelling of battles which begin within the maximum effective ranges of the opposing weapon systems. The parameters m and n allow the fire effectiveness of each side to increase or decrease with time. With $m,n<0$, the fire effectivenesses decrease. This situation might model a battle in which combatants take cover and improve their positions as the battle progresses. With $m,n>0$, the effectivenesses increase, such as might be the case in which two forces are closing with each other.

To model an engagement in which forces close with one another, it is beneficial to relate their instantaneous fire effectiveness to the range separating them. As this range decreases, one might expect fire effectiveness to increase. The attrition-rate coefficients (6) can be expressed in terms of range between forces and the maximum effective ranges of the opposing weapon systems:

$$dx/dt = -a(r)\, y = -a_0 (1-r/R_a)^m y,$$

$$dy/dt = -b(r)\, x = -b_0 (1-r/R_b)^n x. \qquad (10)$$

These equations have been used by S. Bonder [Refs 1 and 3] to model a constant-speed attack on a static defensive position. Figures 1 and 2 show sample relationships between the attrition-rate coefficients and range between opposing forces for various input parameters. In these equations, r is the range between the forces, and R_a and R_b are the

maximum effective ranges of the Y and X weapon systems, respectively. The constants a_0 and b_0 represent the fire effectiveness of each side at r=0. Range and time are related by $r(t)=R_0-vt$, where R_0 denotes the opening range of battle, and v is the constant attack speed. With this relationship, the attrition-rate coefficients in (10) can be related to (7) and (8). For (7) we have:

$$k_a = a_0(v/R_a)^m, \quad k_b = B_0(v/R_b)^n, \qquad C = (R_a-R_0)/v, \tag{11}$$

where $R_a = R_b \geq R_0$, while for (8) we have:

$$k_a = a_0 v/R_a, \; k_b = B_0 v/R_B, \quad A = (R_b-R_a)/v, \quad C = (R_a-R_0)/v, \tag{12}$$

where $R_b \geq R_a \geq R_0$.

IV. A STANDARD FORM FOR THE VARIABLE-COEFFICIENT SOLUTION

The variable attrition-rate coefficient equations (5) yield the force level equations

$$d^2x/dt^2 - [1/a(t)\ da/dt]\ dx/dt - a(t)b(t)\ x = 0, \tag{13}$$

with initial conditions

$$x(t=0) = x_0 \quad \text{and} \quad [1/a(t)\ dx/dt]_{t=0} = -y_0,$$

and similarly

$$d^2y/dt^2 - [1/b(t)\ db/dt]\ dy/dt - a(t)b(t)\ y = 0, \tag{14}$$

with initial conditions

$$y(t=0) = y_0 \quad \text{and} \quad [1/b(t)\ dy/dt]_{t=0} = -x_0.$$

The solution to (13) is given by

$$x(t) = C_1X_1(t) + C_2X_2(t), \tag{15}$$

where $[X_1(t), X_2(t)]$ denotes a fundamental system of solutions to (13), and C_1 and C_2 are constants determined by the initial conditions [see, for example, Ref 7]. The Y-force level has similar form. The functions $X_1(t)$, $X_2(t)$, $Y_1(t)$, and $Y_2(t)$ are referred to by Taylor and Brown as General Lanchester Functions. Their properties are summarized in Table I. The functions $X_1(t)$ and $Y_1(t)$ are similar to the hyperbolic cosine, while $X_2(t)$ and $Y_2(t)$ are similar to the hyperbolic sine. The General Lanchester Functions may be constructed by successive approximations or by infinite series methods.

A. APPLICATION TO POWER ATTRITION-RATE COEFFICIENTS

When this solution method is applied to the case of power attrition-rate coefficients (7), the solution (15) may be written as

$$x(t) = x_0[U_{m,n}(C)u_{m,n}(t+C) - V_{m,n}(C)v_{m,n}(t+C)] -$$
$$y_0\sqrt{k_a/k_b}\,[u_{m,n}(C)v_{m,n}(t+C) - u_{m,n}v_{m,n}(C)], \qquad (16)$$

where $u_{m,n}$, $v_{m,n}$, $U_{m,n}$ and $V_{m,n}$ are referred to by Taylor as Power Lanchester Functions. They have the properties shown in Table II. For computational convenience, utilization is made of the Lanchester-Clifford-Schlafli (LCS) functions

$$F_v(x) = \sum_{k=0}^{\infty}(x/2)^{2k} \Big/ [k\prod_{j=0}^{k-1}(j+v)],$$

$$G_v(x) = \sum_{k=0}^{\infty}(x/2)^{2k+1} \Big/ [k\prod_{j=0}^{k}(j+v)].$$

The Power Lanchester Functions may be expressed in terms of the LCS functions through the following relationships:

$$u_{m,n}(t) = F_q(S(t))$$

$$v_{m,n}(t) = t^{(m-n)/2}G_p(S(t))$$

$$U_{m,n}(t) = F_p(S(t))$$

$$V_{m,n}(t) = t^{(m-n)/2}G_q(S(t)),$$

where $S(t) = \sqrt{(k_a t^m)(k_b t^n)}\ t\,/\,((m+n+2)/2)$. Using the LCS functions, (16) may now be written as

$$x(t) = x_0[F_p(J(0))F_q(J(t)) - (1+t/C)^{(m-n)/2} G_q(J(0))G_p(J(t))$$

$$- y_0\sqrt{a(t=0)/b(t=0)}\,[(1+t/C)^{(m-n)/2} F_q(J(0))G_p(J(t)) -$$

$$G_p(J(0))F_q(J(t))]. \qquad (17)$$

B. APPLICATION TO OFFSET LINEAR COEFFICIENTS

Similar techniques are utilized to apply this general theory to the case of the offset linear attrition-rate coefficients (8). With these coefficients, a fundamental system of solutions to (13) is given by

$$x_1(t)=f(t+C),\quad x_2(t)=g(t+C),\quad y_1(t)=F(t+C),\quad y_2(t)=G(t+C)$$

where

$$f(t) = \sum_{n=0}^{\infty}\{(\sqrt{k_a k_b}/2)^{2n}/(2n)!\}\sum_{k=0}^{n} B_n^k A^k t^{4n-k}$$

$$g(t) = \sum_{n=0}^{\infty}\{(\sqrt{k_a k_b}/2)^{2n+1}/(2n+1)!\}\sum_{k=0}^{n} C_n^k A^k t^{4n+2-k}$$

$$F(t) = \sum_{n=0}^{\infty}\{(\sqrt{k_a k_b}/2)^{2n}/(2n)!\}\sum_{k=0}^{n} D_n^k A^k t^{4n-k}$$

$$G(t) = \sum_{n=0}^{\infty}\{(\sqrt{k_a k_b}/2)^{2n+1}/(2n+1)!\}\sum_{k=0}^{n+1} E_n^k A^k t^{4n+2-k}.$$

The coefficients B_n^k, C_n^k, D_n^k, and E_n^k satisfy complicated recurrence relations [see Taylor and Brown (Ref 13)]. The functions f(t), g(t), F(t), and G(t) are referred to as Offset Linear Lanchester Functions. As in the case of the power attrition-rate coefficients, use is made of the following Auxiliary Offset Linear Lanchester Functions for computational convenience:

$$h(p,u) = \sum_{n=0}^{\infty} \{p^{2n}/(2n)!\} \sum_{k=0}^{n} B_n^k p^k$$

$$w(p,u) = \sum_{n=0}^{\infty} \{p^{2n+1}/(2n+1)!\} \sum_{k=0}^{n} C_n^k u^k$$

$$H(p,u) = \sum_{n=0}^{\infty} \{p^{2n}/(2n)!\} \sum_{k=0}^{n} D_n^k u^k$$

$$W(p,u) = \sum_{n=0}^{\infty} \{p^{2n+1}/(2n+1)!\} \sum_{k=0}^{n+1} E_n^k u^k .$$

Utilizing these Auxillary Lanchester Functions, the X-force level solution (15) may be written as

$$\begin{aligned} x(t) = {} & x_0[H(J(0),j(0))\ h(J(t),j(t)) - W(J(0),j(0)) \\ & w(J(t),j(t))] \\ & - y_0\sqrt{k_a/k_b}[h(J(0),j(0))\ w(J(t),j(t)) - w(J(0),j(0)) \\ & h(J(t),j(t))], \end{aligned} \tag{18}$$

where $j(t) = A/(t+C)$ and $J(t) = \sqrt{k_a k_b}\ (t+C)^2/2$.

The equations (17) and (18) are solutions to the X-force level equation (13) for the case of power and offset linear attrition-rate coefficients, respectively. This sclution method of Taylor and Brown represents a considerable extension of Lanchester theory in the area of variable attrition-rate coefficients. Previous analytic results had been available only in the more limited case of the opening range of battle equal to the minimum cf the maximum effective ranges of the two weapon systems. That is, the initial fire effectiveness of at least one side was restricted to be zero [see for example Ref 12].

While the time solutions (17) and (18) appear formidable, their implementation is simplified if a digital computer is available. As noted earlier in this paper,

however, these solutions are complex and involve new mathematical functions. This makes it difficult for the military analyst to perceive significant relationships among battle variables, and renders parametric analysis difficult and time-consuming. This provides motivation for the approximation technique to follow.

V. REDUCTION TO LIOUVILLE'S NORMAL FORM AND APPROXIMATION

The argument ab (referred to by, among others, Taylor and Brown [Ref 13] as the "intensity of combat") in the constant coefficient result (4) provides motivation for the transformation [Ref 14]

$$z = \int_{t_0}^{t} \sqrt{a(s)b(s)}\, ds, \tag{19}$$

where $z(t=0)$ is denoted z_0. Applying this transformation to (13) and (14) yields

$$d^2x/dz^2 + \{1/2\ d/dz\ \ln[b(t)/a(t)]\}dx/dz - x = 0, \tag{20}$$

with initial conditions

$$x(z=z_0) = x_0 \quad \text{and} \quad \{[b(t)/a(t)]^{1/2}\ dx/dz\}_{z=z_0} = -y_0,$$

and

$$d^2y/dz^2 + \{1/2\ d/dz\ \ln[a(t)/b(t)]\}\ dy/dz - y = 0, \tag{21}$$

with initial conditions

$$y(z=z_0) = y_0 \quad \text{and} \quad \{[a(t)/b(t)]^{1/2}\ dy/dz\}_{z=z_0} = -x_0.$$

Taylor and Brown have shown that the X force level equation (13) may be transformed into a linear second order differential equation with constant coefficients if and only if (20) is a constant coefficient equation. This is the case if and only if

$$(1/\sqrt{a(t)b(t)})\ d/dt\ \ln[a(t)/b(t)] = \text{CONSTANT}.$$

Equation (20) shows the significance of the parameters $a(t)/b(t)$ (the relative effectiveness of the two weapon

systems), and the intensity of combat $\sqrt{a(t)b(t)}$ in determining battle outcome. This is a generalization of the well-known constant coefficient results to the case of temporal variations in fire effectiveness. The significance of these parameters may be seen more explictly by transformation to Liouville's normal form (see Ref 14).

The transformation

$$x(z) = X(z)[(a(t)/a_0) / (b(t)/b_0)]^{1/4}, \tag{22}$$

may be applied to (20) and yields the normal form with the first derivative of the dependent variable removed

$$d^2X/dz^2 - [1 + F(z)]X = 0, \tag{23}$$

with initial conditions

$$X(z=z_0) = x_0 \quad \text{and} \quad dX/dz(z=z_0) = -y_0\sqrt{a_0/b_0} - x_0e_0,$$

where

$$F(z) = P''(z) / P(z) \qquad P(z) = [R(t)]^{-1/4} \tag{24}$$

$$R(t) = a(t)/b(t) \qquad e(t) = 1/4\sqrt{a(t)b(t)}\ d/dt\ \ln R(t). \tag{25}$$

In (24), the notation $P''(z)$ denotes d^2P/dz^2. Equation (23) is Liouville's normal form [see page 23 of Ref 9].

For the Y force level, the transformation

$$y(z) = Y(z)[(b(t)/b_0) / (a(t)/a_0)]^{1/4},$$

yields the normal form

$$d^2Y/dz^2 - [1 + G(z)]Y = 0, \tag{26}$$

with initial conditions

$$Y(z=z_0) = y_0 \quad \text{and} \quad dY/dz(z=z_0) = -x_0\sqrt{b_0/a_0} + y_0 e_0,$$

where

$$G(z) = Q''(z) \,/\, Q(z) \quad \text{and} \quad Q(z) = [R(t)]^{1/4} = 1/P(z).$$

Expressing the transformed force-level equation (23) in the simple form

$$d^2X/dz^2 - X = Q(z)X,$$

variation of parameters may be used to obtain the solution to (23) as

$$X(z) = x_0 \cosh(z-z_0) - [y_0\sqrt{a_0/b_0} + x_0 e_0] \sinh(z-z_0) + \int_{z_0}^{z} F(j) \sinh(z-j) X(j)\, dj. \tag{27}$$

This equation may be expressed in terms of the original time variable t and dependent variable x as

$$x(t) = [(a(t)/a_0)/(b(t)/b_0]^{1/4} \{x_0 \cosh(\int_0^t \sqrt{a(s)b(s)}\,ds) - [y_0\sqrt{a_0/b_0} + x_0 e_0] \sinh(\int_0^t \sqrt{a(s)b(s)}\,ds)\}$$

$$+ [a(t)/b(t)]^{1/4} \, d/ds\{1/(a(s)b(s))\, d/ds[b(s)/a(s)]^{1/4}\} \sinh(\int_s^t \sqrt{a(sj)b(j)}\,dj)\, X(s)\, ds. \tag{28}$$

If the relative effectiveness varies slowly, then the relationship (24) would indicate that $F(z) \ll 1$ so that one could drop the integral term in (27) yielding

$$X(z) = x_0 \cosh(z-z_0) - [y_0\sqrt{a_0/b_0} + x_0 e_0] \sinh(z-z_0), \tag{29}$$

which is the Liouville-Green approximation [Ref 11]. The approximation may be expressed in terms of the original variables t and x as

$$x(t) = [(a(t)/a_0)/b(t)/b_0]^{1/4} \{x_0 \cosh(\int_0^t \sqrt{a(s)b(s)}\,ds) - [y_0\sqrt{a_0/b_0} + x_0 e_0] \sinh(\int_0^t \sqrt{a(s)b(s)}\,ds)\} \qquad (30)$$

VI. APPLICATION TO POWER AND OFFSET LINEAR COEFFICIENTS

The Liouville-Green approximation may now be applied to the constant-speed attack model with attrition-rate coefficients (7). For the reader's convenience, these coefficients are

$$a(t) = k_a (t+C)^m \quad \text{and} \quad b(t) = k_b (t+C)^n. \tag{7}$$

This model assumes that the maximum effective ranges of the two opposing weapon systems are equal (no offset). The opening range of battle is taken to be within this maximum effective range (i.e., C>0). Applying the Liouville-Green approximation (29), the time solution to the X force level equation in terms of the original time variable t and dependent variable X becomes

$$x(t) = (1+t/c)^{(m-n)/4} \{x_0 \cosh \int_0^t \sqrt{a(s)b(s)}\, ds$$

$$- [y_0 \sqrt{k_a/k_b}\, C^{(m-n)/2} + (x_0 (m-n)/4 \; k_a k_b) C^{-(m+n+2)/2}]$$

$$\sinh \int_0^t \sqrt{a(s)b(s)}\, ds\}, \tag{31}$$

where

$$\int_0^t \sqrt{a(s)b(s)}\, ds = (\sqrt{k_a k_b}/(m+n+2)/2)$$

$$\{(t+C)^{(m+n+2)/2} - C^{(m+n+2)/2}\}.$$

For future reference, equation (24) becomes in this case

$$F(z) = [(m-n)(3m+n+4)] / [4(m+n+2)^2 z^2], \tag{32}$$

or in terms of the original time variable t and range parameters (11):

$$F(t) = [R_a^m R_b^n (m-n)(3m+n+4)] /$$

$$[16 a_0 b_0 (vel)^{m+n} (t+(R_a - R_0)/vel)^{m+n+2}], \qquad (33)$$

where vel = constant velocity of attack.

We can also apply the Liouville-Green approximation to the case of the offset linear attrition-rate coefficients

$$a(t) = k_a(t+C) \quad \text{and} \quad b(t) = k_b(t+C+A). \qquad (8)$$

In this case, the time solution to the X force level equation becomes

$$x(t) = [(1+(A/C)/(1+(A/(t+C)))]^{1/4} \{x_0 \cosh \int_0^t \sqrt{a(s)b(s)}\,ds$$

$$- [(y_0 \sqrt{k_a/k_b})/(1+(A/C)) + (x_0 (A/C))/$$

$$(4\, k_a k_b C^2 (1+(A/C))^{3/2})] \sinh \int_0^t \sqrt{a(s)b(s)}\,ds\}, \qquad (34)$$

where

$$\int_0^t \sqrt{a(s)b(s)}\,ds = (A^2 \; k_a k_b/8)\{(1+(2(t+C)/A)) \; (1+(2(t+C)/A))^2 - 1$$

$$- \ln[1+(2(t+C)/A) + (1+(2(t+C)/A))^2 - 1]\}$$

For future reference, equation (24) becomes in this case

$$F(t) = R_a R_B (R_B - R_a)[12(t+(R_a - R_0)/vel) + 7((R_B - R_a)/vel)] /$$

$$\{16 a_0 b_0 vel^3 (t+(R_a - R_0)/vel)^3 (t+(R_b - R_0)/vel)^3\}. \qquad (35)$$

VII. DISCUSSION

A computer program was developed in FORTRAN to produce plots of the X force level versus range between opposing forces for the constant-speed attack model described earlier. Both the power and offset linear attrition-rate coefficient cases (equations (31) and (34)) were considered. These approximate solutions were compared to the exact results given by Taylor and Brown [Ref 13] to evaluate the accuracy of the Liouville-Green approximation to equations with these coefficients.

A. POWER ATTRITION-RATE COEFFICIENTS

For the case of power attrition-rate coefficients, the maximum effective ranges of the two weapon systems were set equal to 2000 meters, with an opening range of battle of 1250 meters. The fire effectiveness of the two sides at zero range separation was a_0=.06 X casualties/min-Y unit, and b_0=.6 Y casualties/min-X unit. The velocity of attack was constant at five miles per hour, and the initial force levels were X = 10 and Y = 30. These parameter values correspond to those used by Taylor and Brown in their paper [Ref 13]. The exponents m and n in (7) were varied throughout the range explored in Ref 13. The results are shown in Figures 3 through 10, where each figure contains the approximate time solution and the corresponding exact analytical solution obtained by Taylor and Brown. As even a cursory examination reveals, the approximation appears to conform to the established results only in certain cases. Some possible explanations for this observed deviation will now be considered.

Recall that the integral term in equation (27) was

dropped (assumed to be approximately zero) in making the Liouville-Green approximation. The validity of this assumption appears to be linked strongly to the form of F(z), which for the present case is

$$F(t) = [R_a^m R_b^n (m-n)(3m+n+4)] \, / \, [16a_0 b_0 (vel)^{m+n} (t+(R_a-R_0)/vel)^{m+n+2}]. \tag{33}$$

When the exponents m and n are equal, F(t)=0, and by considering equation (27), one can see that the approximation is, in fact, the exact solution. This is verified trivially by Figures 3 and 4, for which the exponents are equal. It should be noted that in this situation, the model reduces to the case of a constant ratio of attrition-rate coefficients.

When the exponent m is greater than n, F(t) will always be greater than zero, and from (27) it follows that the exact solution is greater than the approximation. This situation is illustrated by Figures 6 and 10. The converse also holds. When m is less than n, F(t) is always less than zero, and the approximation is greater than the exact solution. This is shown in Figures 5 and 9.

Again examining (33), the parameter $|a_0 b_0|$, which is the product of the opposing forces' fire effectivenesses at r=0, appears to be significant. As $|a_0 b_0|$ increases, F(t) decreases, making the integral term in (27) smaller, so one would expect the approximation to be "closer" to the exact solution. This did occur in the cases considered (see for example Figure 7).

B. OFFSET LINEAR COEFFICIENTS

For the case of the offset linear attrition-rate coefficients (8), the maximum effective range of the Y force weapon system (R_a) was set equal to 1500 meters, with the rest of the parameter values equal to those in the power coefficient case. The maximum effective range of the X force weapon system (R_b) was varied throughout the range considered in Ref 13. The results are shown in Figures 11 through 15 where, as before, each figure compares the corresponding solutions. As was the case with the power coefficients, the solutions are seen to conform only in certain cases.

For the linear coefficients, it was shown that F(z) could be written in terms of the time variable t as

$$F(t) = R_a R_B (R_B - R_a)[12(t+(R_a-R_0)/vel) + 7((R_B-R_a)/vel)] /$$

$$\{16a_0 b_0 vel^3 (t+(R_a-R_0)/vel)^3 (t+(R_b-R_0)/vel)^3\}. \quad (35)$$

When the maximum effective ranges of the opposing weapon systems (R_a and R_b) are equal, $F(t) = 0$, and the integral term in (27) is zero. This again reduces to the constant attrition-rate-ratio case as can be seen from (8) with A=0. Thus one would again expect agreement of the exact and approximate solutions. This was found to be true as shown by Figure 11, where both maximum effective ranges equal 1500 meters.

As was the case for the power coefficients, the parameter $|a_0 b_0|$ appears in the denominator of F(t), hence the two solutions should be closer as $|a_0 b_0|$ increases. This

was found to be the case, as illustrated by Figure 15.

C. ANALYTICAL CONSIDERATIONS

The examinations of both the power and offset linear attrition-rate coefficient cases have centered on the parameter F(z) in the solution (27). While F(z) does appear to be the driving factor in the integral term in (27) (thus in the validity of the approximation), it should be made clear that the other factors in the integral term can not be ignored. While a mathematical analysis(for example of the hyperbolic sine term) would be formidible, examples can be found which illustrate that F(z) can not always be considered alone. One such example is shown in Figure 8. The fire effectiveness of the two forces at r=0 were interchanged (i.e., a_0=.6 and b_0=.06 instead of vice versa) in the power attrition-rate coefficient case. From the appropriate form of F(z), equation (33), one can see that this change has no effect on the numerical value of F(t) throughout the time range. Yet the significant difference in results can be seen in Figure 8 when compared to Figure 6. Currently, there is no convenient methodology to analyze the implications of such discrepancies.

Toward this end, theoretical analysis in the area of Olver's "Error Bounds for the Liouville-Green Approximation" [Ref 11] could be very beneficial. While they sound promising, Olver's results can not be directly applied to variable-coefficient Lanchester-type differential equations. If available in a convenient form, such error bounds could be evaluated for specific situations to investigate tolerable error limits prior to invoking the approximation.

From the above analytic considerations, one can see

that while one may be able to explain some of the error incurred, this is small comfort to the analyst attempting to make practical use of the Liouville-Green approximation. It has been shown that the approximation appears to yield credible results only in certain limited circumstances. Furthermore, methodology is not presently available to predict the error which will be incurred in a specific situation, except for the trivial case in which the integral term in (27) goes to zero.

VIII. SUMMARY

This thesis has examined the "adequacy" of the so-called Liouville-Green approximation to the solution of variable-coefficient Lanchester-type equations of modern warfare. Although this approximation only involves "elementary" functions and has an intuitively appealing form, it was unfortunately found that this Liouville-Green-Lanchester approximation is not consistently reliable for estimating force levels. Specifically, the approximation was applied to two forms of attrition-rate coefficients: (I) power attrition-rate coefficients (modelling, for example, the same maximum effective ranges for the opposing weapon systems), and (II) offset linear attrition-rate coefficients (modelling different maximum effective ranges). It was seen that the approximation was made by dropping the integral term in equation (27), hence exact results are achieved only when this term equals zero. This situation was found to occur only when the ratio of attrition-rate coefficients was constant. For the power attrition-rate coefficients, this implies that the exponents in (7) are equal, while for the offset linear coefficients this implies the offset parameter A in (8) is zero (which occurs when the maximum effective ranges are equal). Analytic considerations showed that while certain initial parameter combinations produced credible results from the approximation, there was no methodology available to predict the error incurred in the general case. It was found that, as a general rule, the further one moves from the constant coefficient ratio case, the larger the error incurred by invoking the Liouville-Green approximation.

TABLE I.

Properties of the General Lanchester Functions x_1, x_2, y_1, y_2.

1. $dx_1/dt = \sqrt{k_b/k_a}\, a(t) y_2$

2. $dx_2/dt = \sqrt{k_b/k_a}\, a(t) y_1$

3. $dy_1/dt = \sqrt{k_a/k_b}\, b(t) x_2$

4. $dy_2/dt = \sqrt{k_a/k_b}\, b(t) x_1$

5. $x_1(t) y_1(t) - x_2(t) y_2(t) = 1 \qquad \forall t$

Table II. Properties of the Power Lanchester Functions

$u_{m,n}$, $v_{m,n}$, $U_{m,n}$, and $V_{m,n}$.

1. $du_{m,n}/dt = \sqrt{k_b/k_a}\,[k_a t^m]V_{m,n}(t)$

2. $dv_{m,n}/dt = \sqrt{k_b/k_a}\,[k_a t^m]U_{m,n}(t)$

3. $dU_{m,n}/dt = \sqrt{k_a/k_b}\,[k_b t^n]v_{m,n}(t)$

4. $dV_{m,n}/dt = \sqrt{k_a/k_b}\,[k_b t^n]u_{m,n}(t)$

5. $u_{m,n}(t)U_{m,n}(t) - v_{m,n}(t)V_{m,n}(t) = 1 \quad \forall t$

6. $u_{m,n}(t=0) = U_{m,n}(t=0) = 1$

7. $v_{m,n}(t=0) = V_{m,n}(t=0) = 0$

8. $u_{m,m}(t) = U_{m,m}(t) = \cosh\,(\sqrt{k_a k_b}\; t^{m+1}/(m+1))$

9. $v_{m,m}(t) = V_{m,m}(t) = \sinh\,(\sqrt{k_a k_b}\; t^{m+1}/(m+1))$

Properties 6 and 7 only hold for $m,n > -1$.

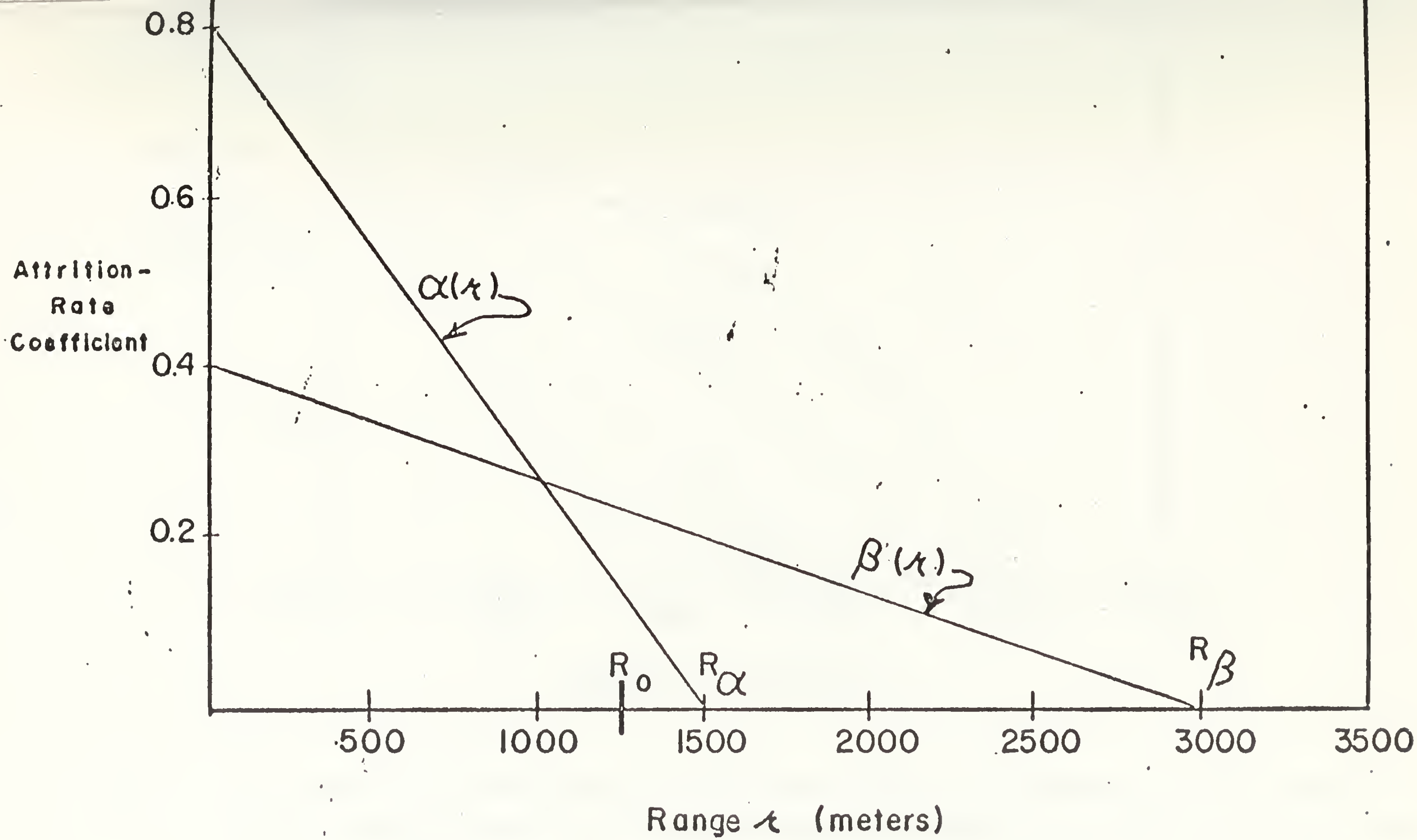

Figure 1. Linear attrition-rate coefficients for weapon systems with different effective ranges. [Notes: 1. The maximum effective ranges of the two weapon systems are denoted as R_α and R_β. 2. The opening range of battle is denoted as R_0 and (as shown) $R_0 <$ minimum (R_α, R_β).]

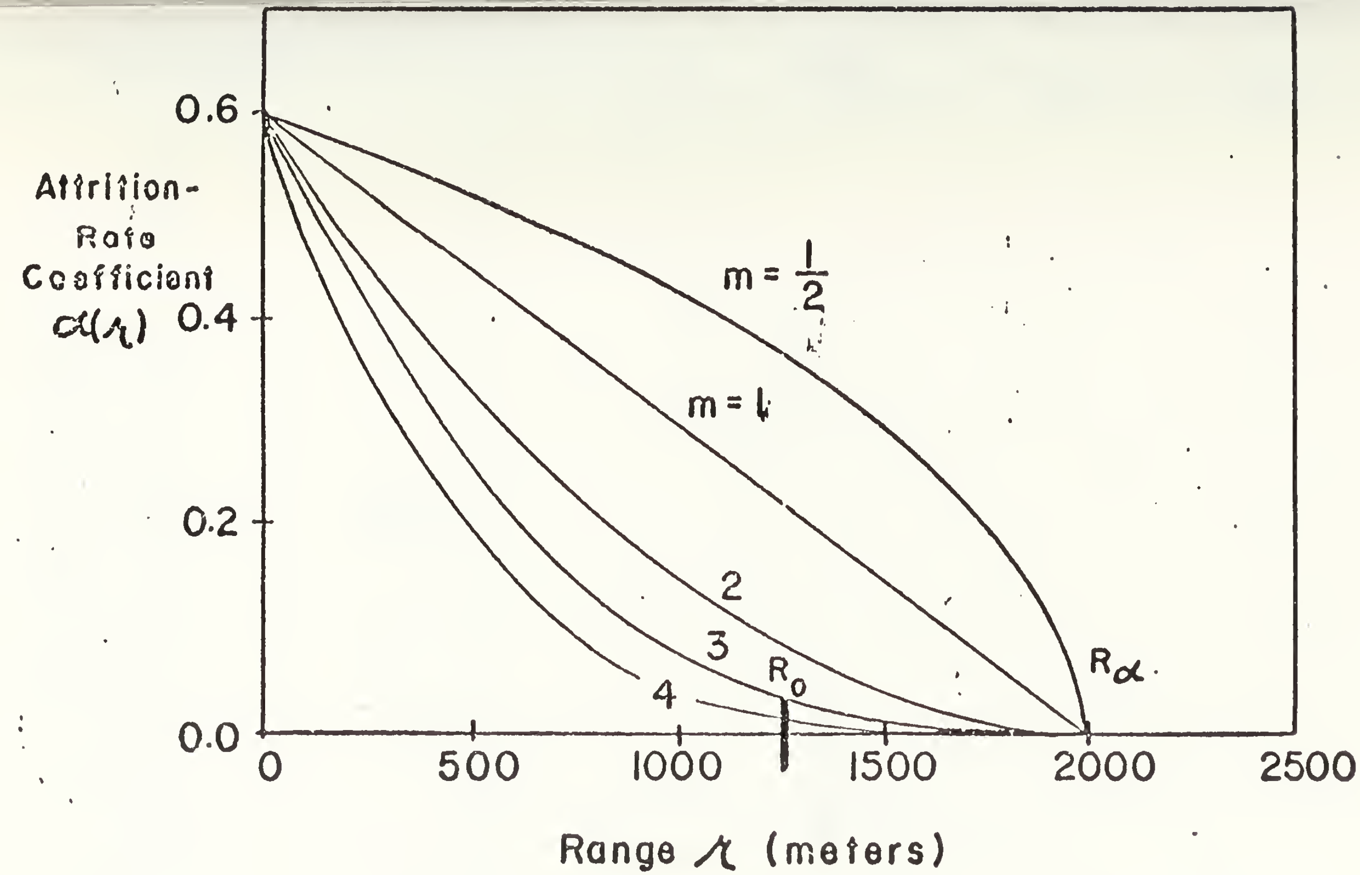

Figure 2. Dependence of the attrition-rate coefficient $\alpha(r)$ on the exponent m for constant maximum effective range of the weapon system and constant kill capability at zero range. [Notes: 1. The maximum effective range of the system is denoted $R_\alpha = 2000$ meters. 2. $\alpha(r=0) = \alpha_0 = 0.6$ X casualties/(unit time × number of Y units) denotes the Y force weapon system kill rate at zero force separation (range). 3. The opening range of battle is denoted as $R_0 = 1250$ meters and (as shown) $R_0 < R_\alpha$.]

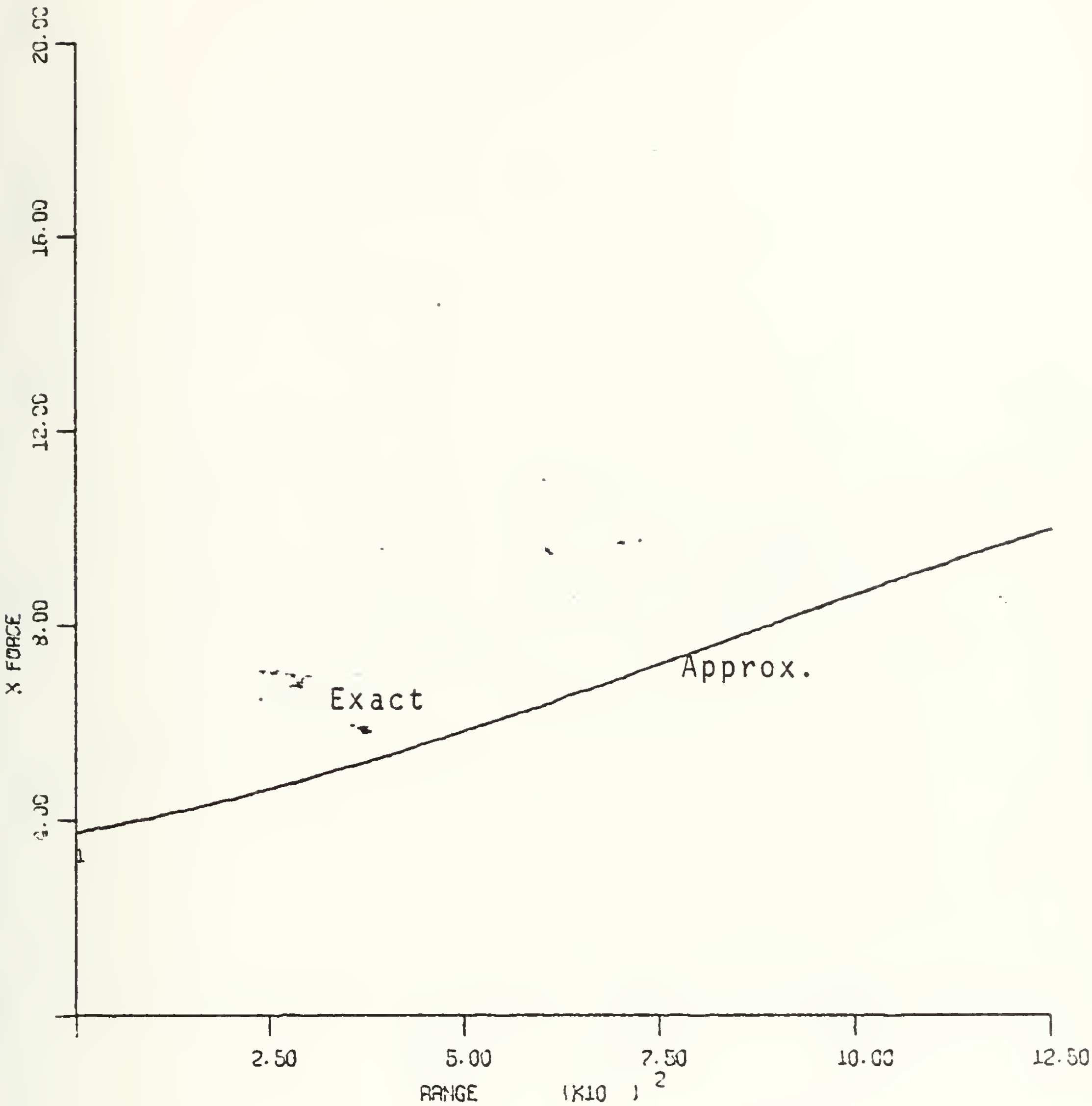

Figure 3. Power coefficients with m=n=1.
For Figures 3 through 10, $R_a=R_b=2000$m., $x_0=10$, $y_0=30$, $a_0=.06$, and $b_0=.6$ unless otherwise specified.

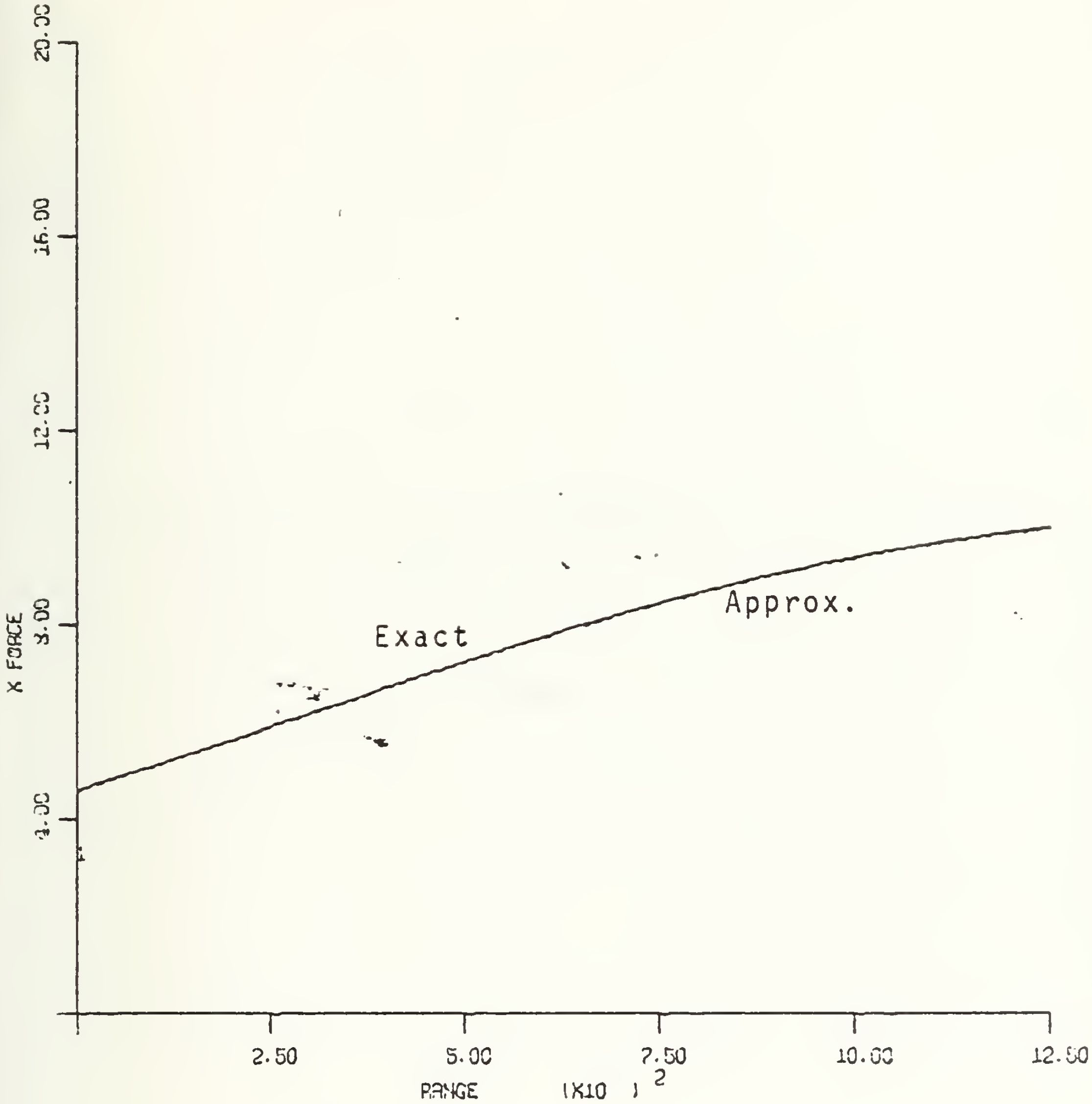

Figure 4. Power coefficients with m=n=2.

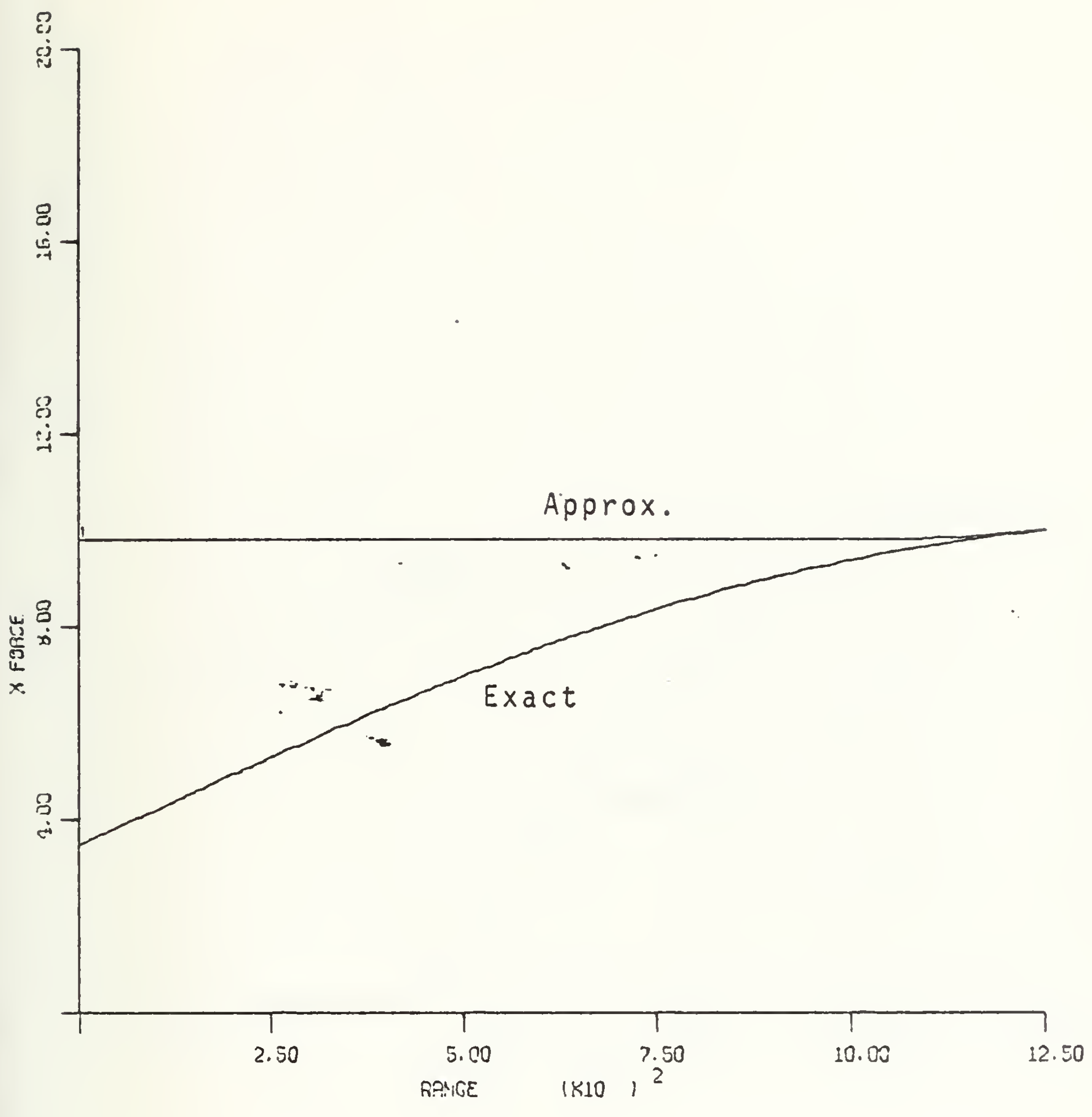

Figure 5. Power coefficients with m=2, n=3.

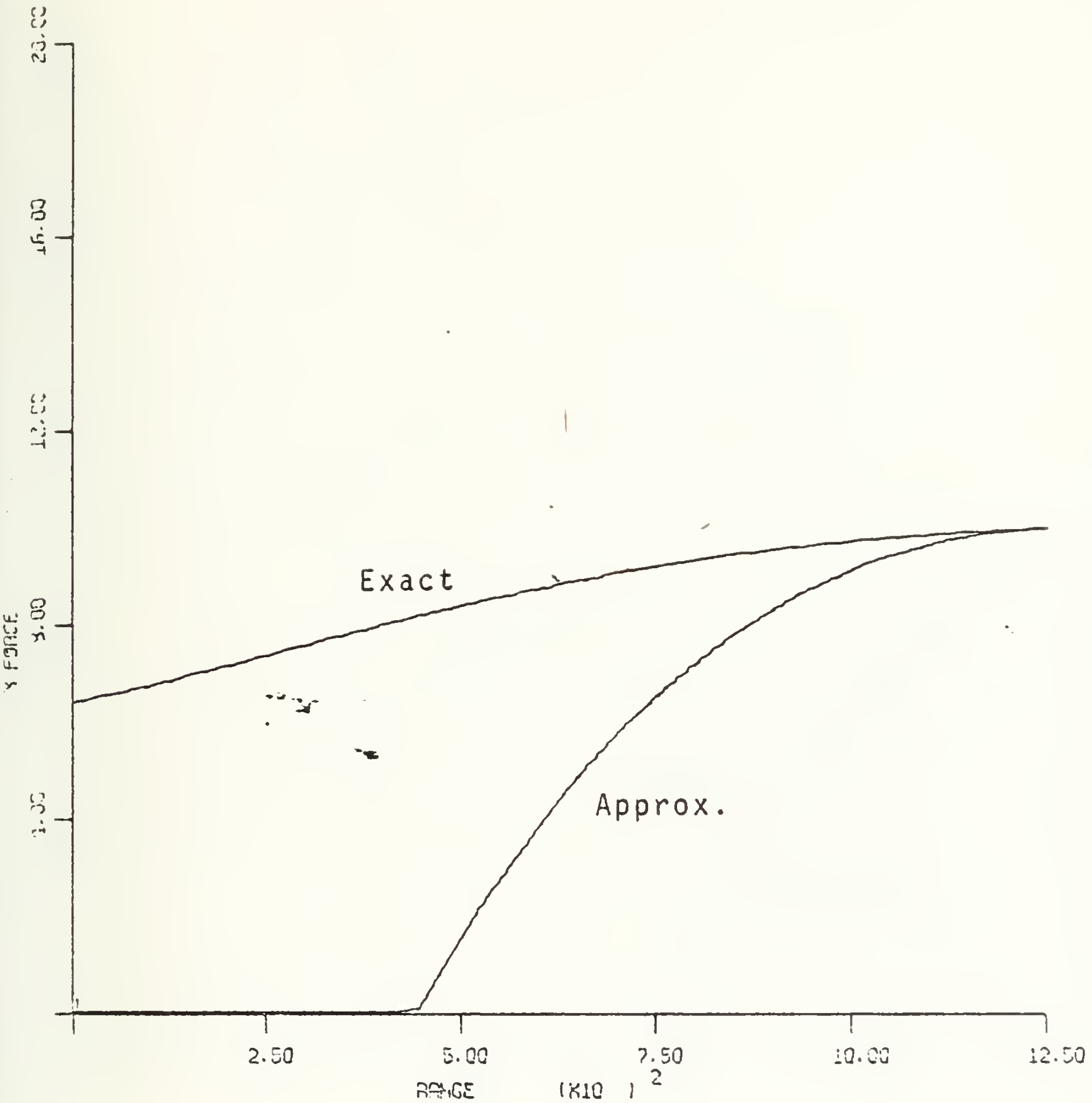

Figure 6. Power coefficients with m=3, n=2.

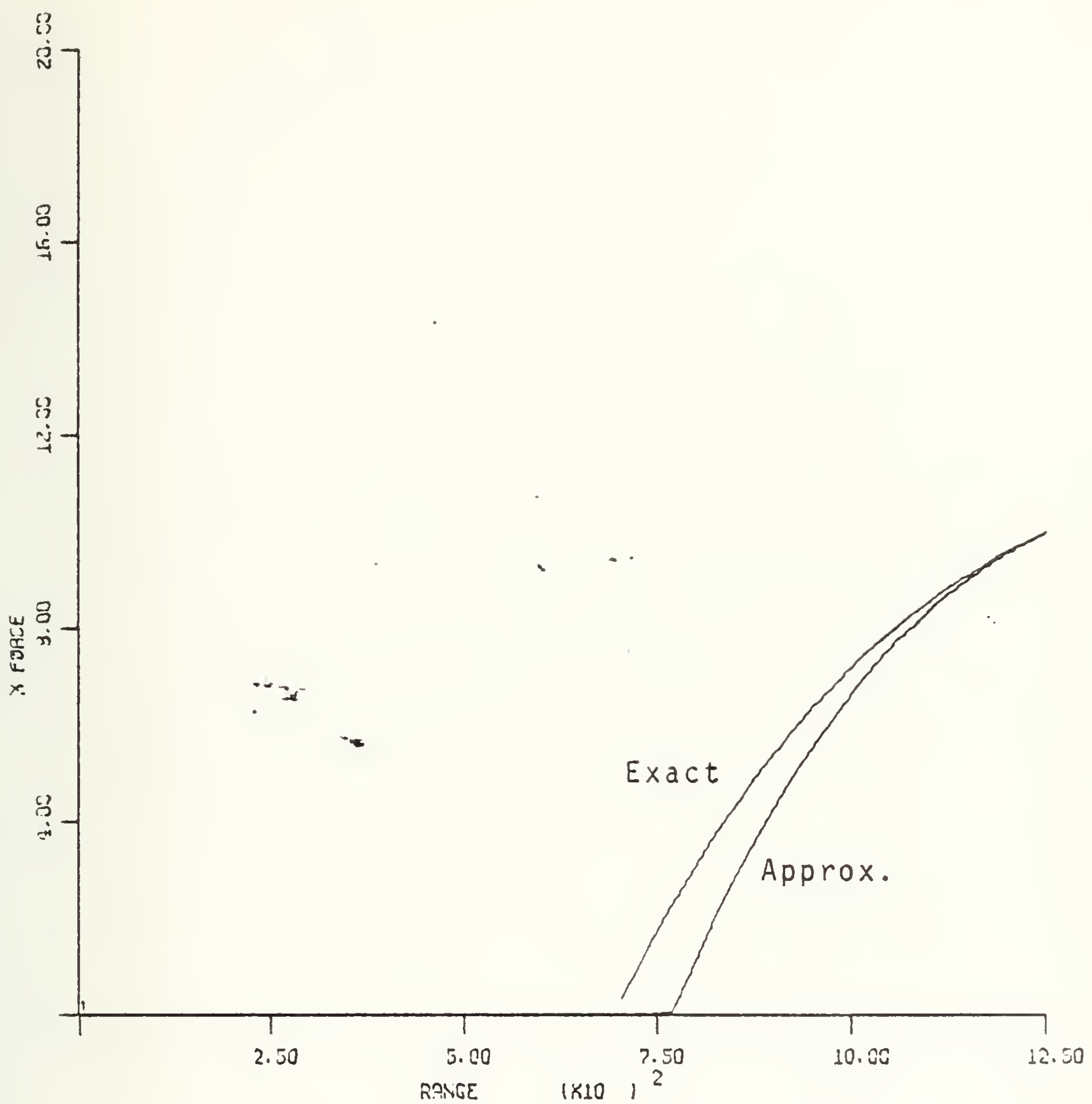

Figure 7. Power coefficients with m=3, n=2, $a_0 = B_0 = .6$.

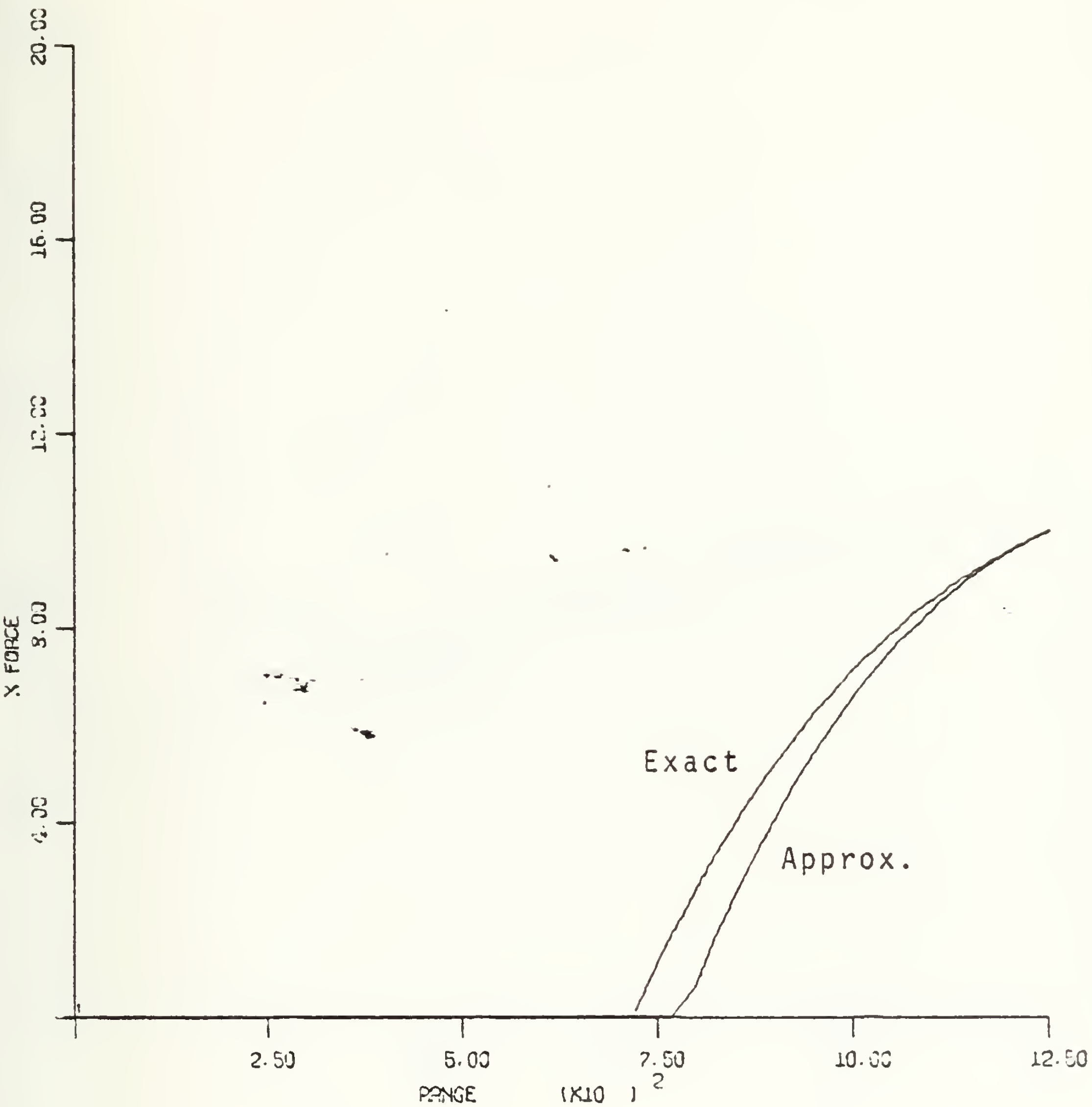

Figure 8. Power coefficients with m=3, n=2, a_0 = .6, B_0 = .06

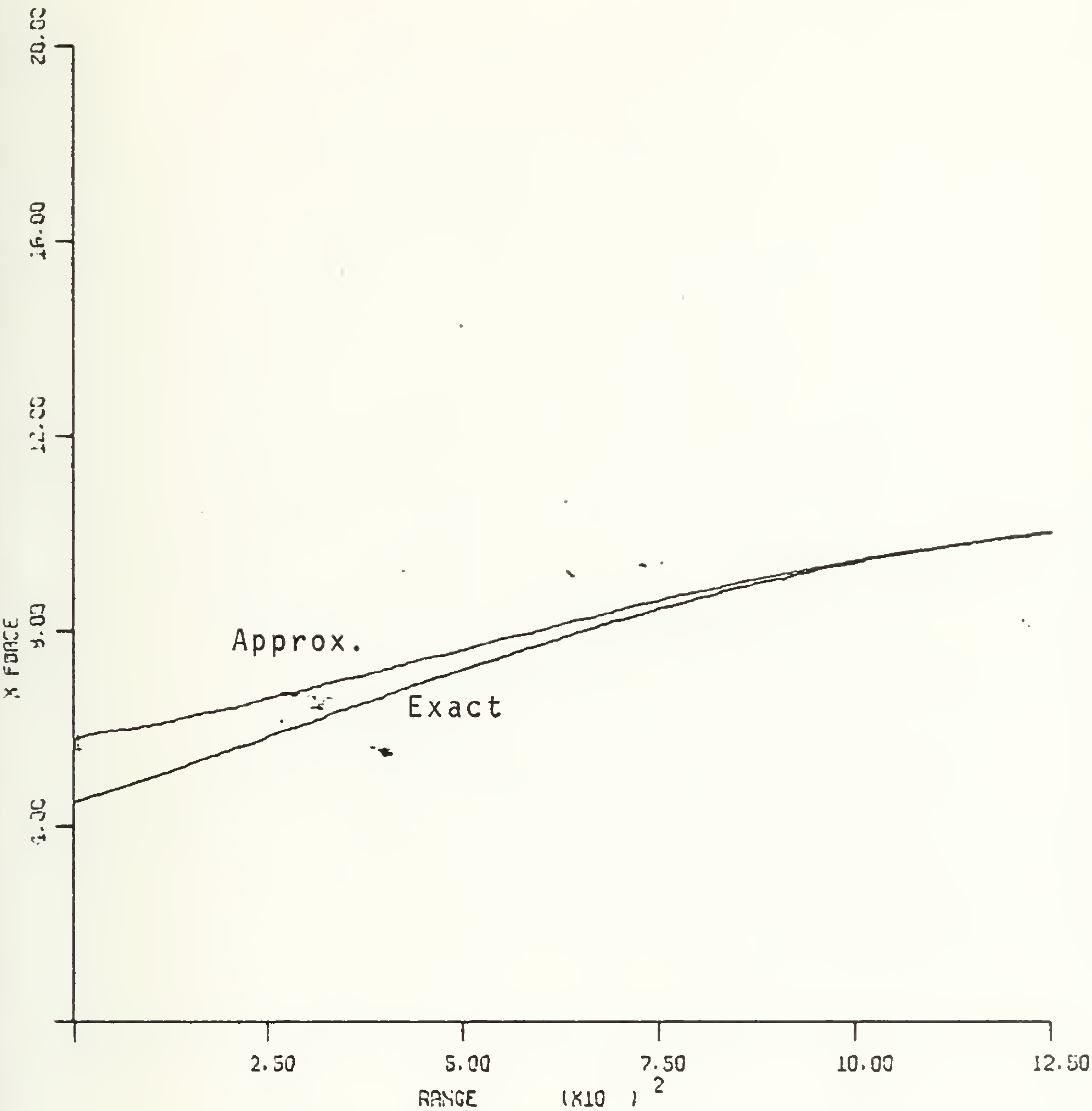

Figure 9. Power coefficients with m=2, n=2.1.

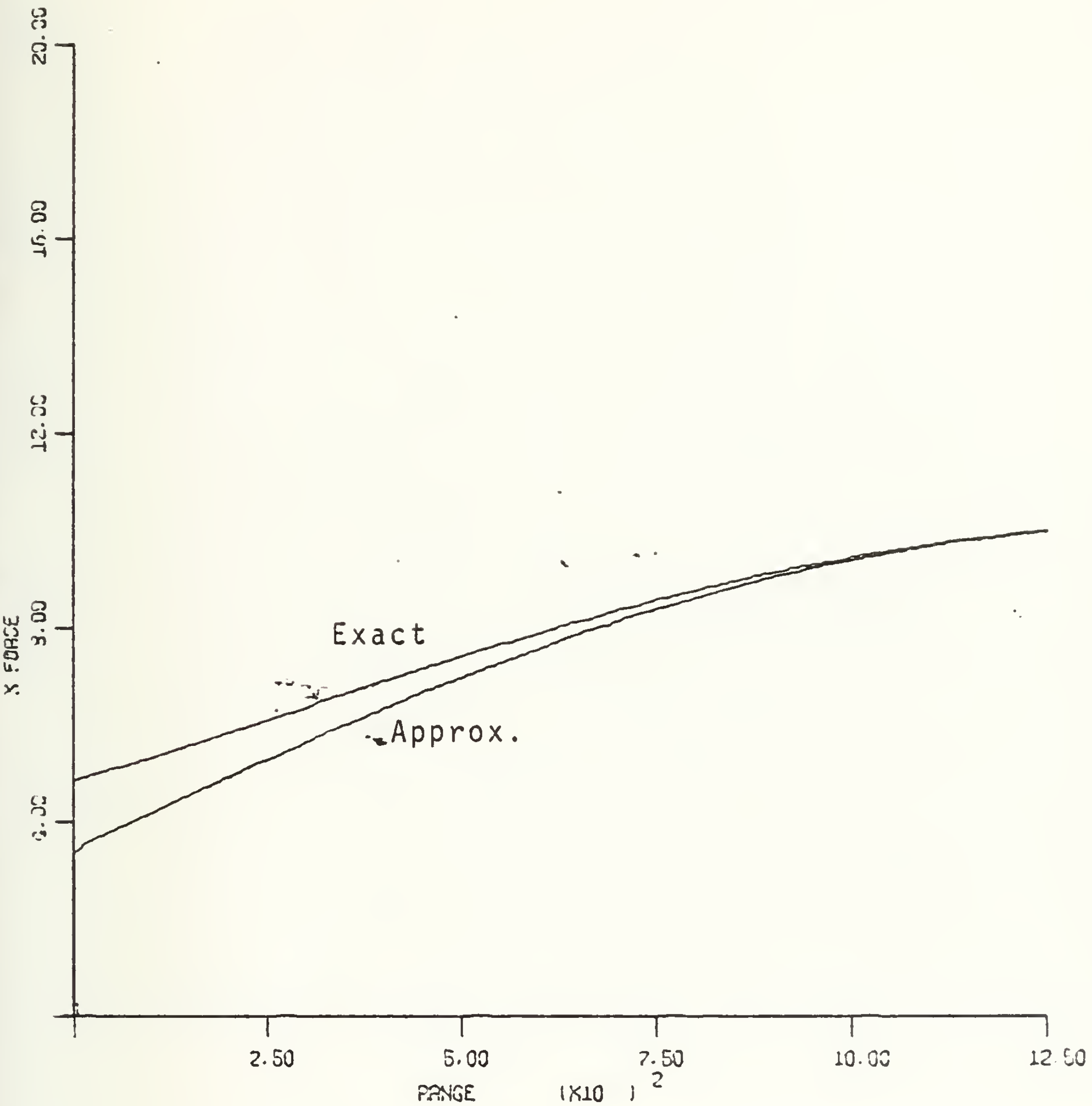

Figure 10. Power coefficients with m=2.1, n=2.

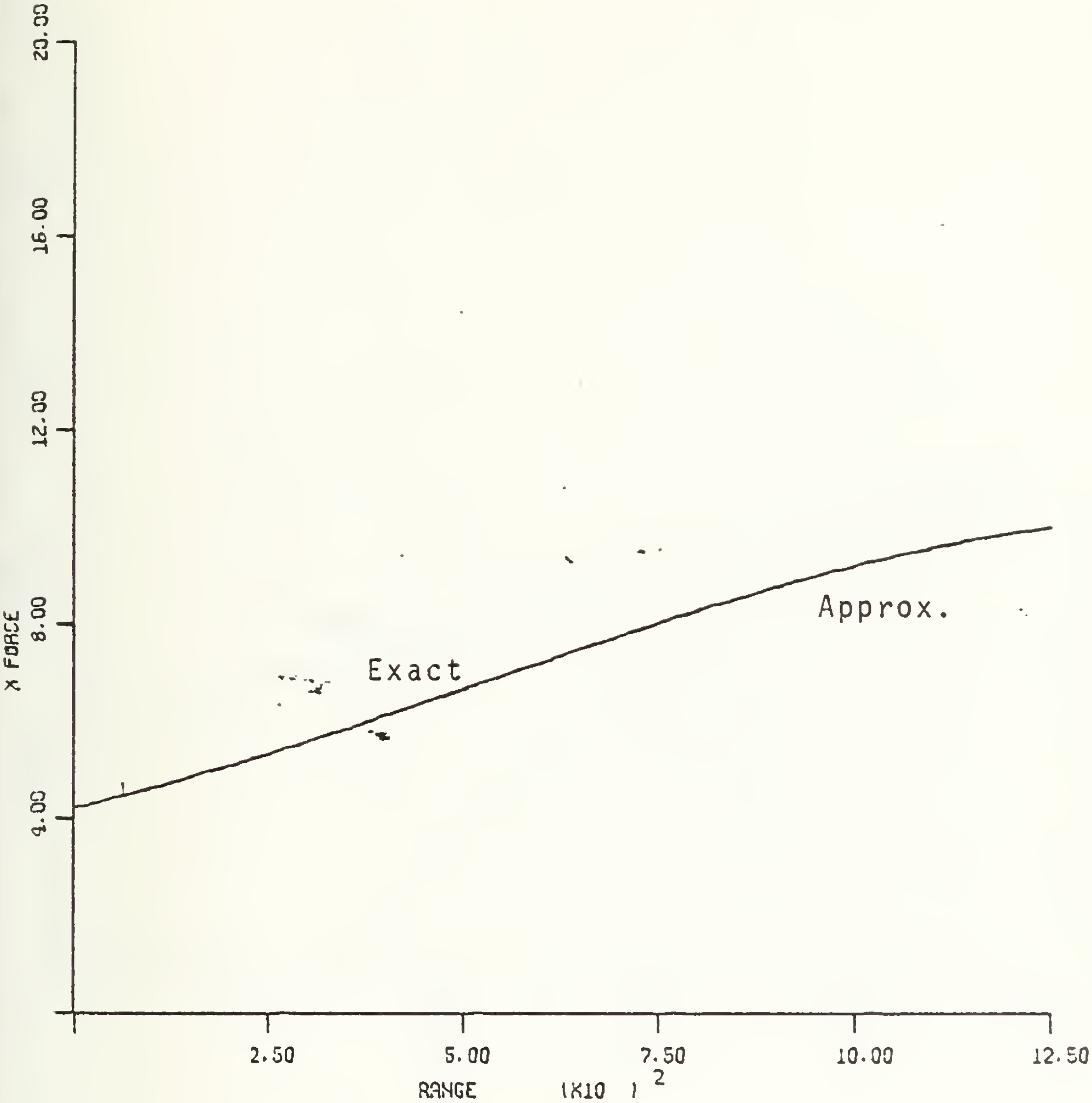

Figure 11. Linear coefficients with R_B = 1500m. For Figures 11 through 16, R_a=1500m., x_0=10, y_0=30, a_0=.06, and b_0=.6 unless otherwise specified.

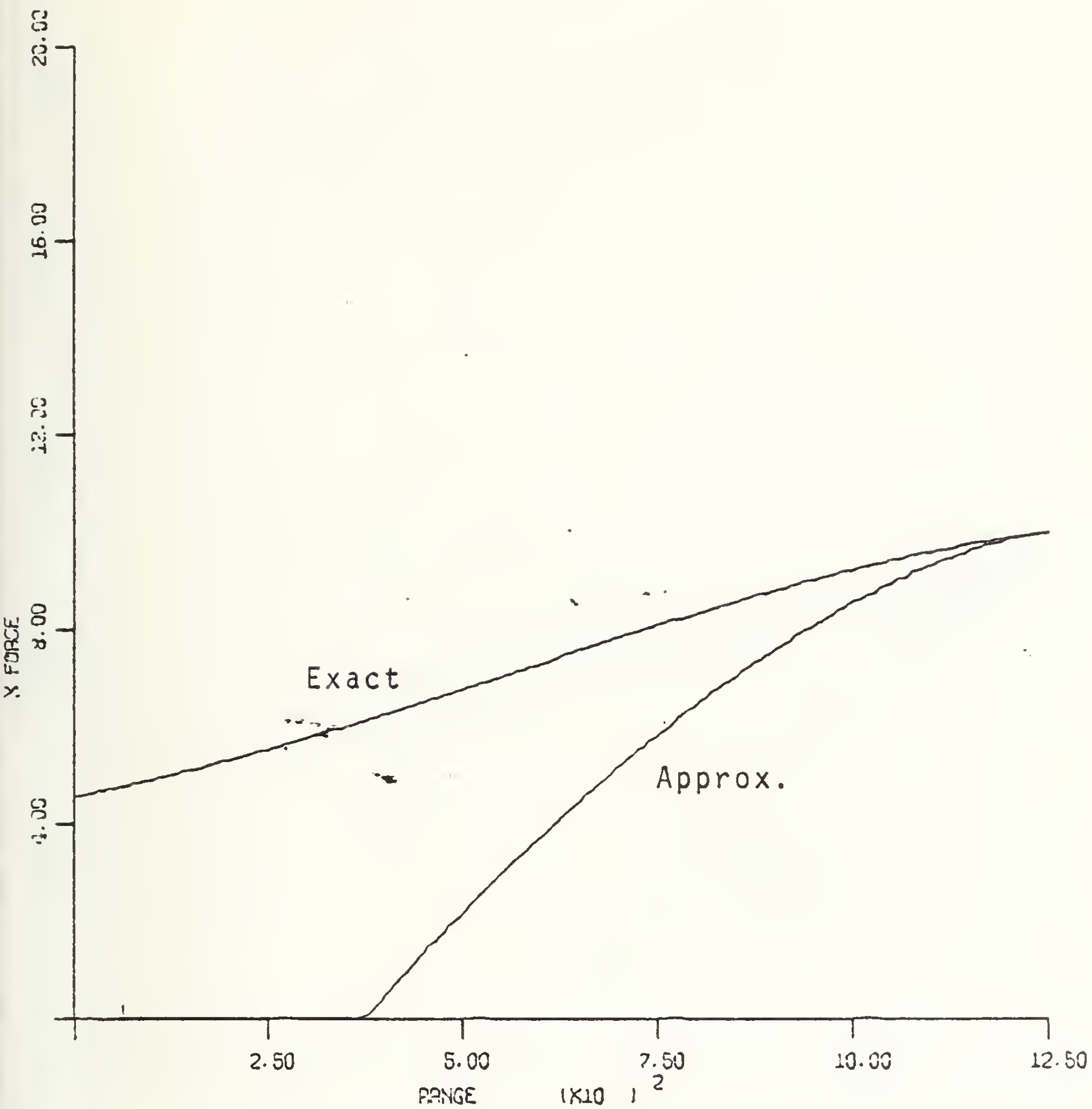

Figure 12. Linear coefficients with R_B = 1600m.

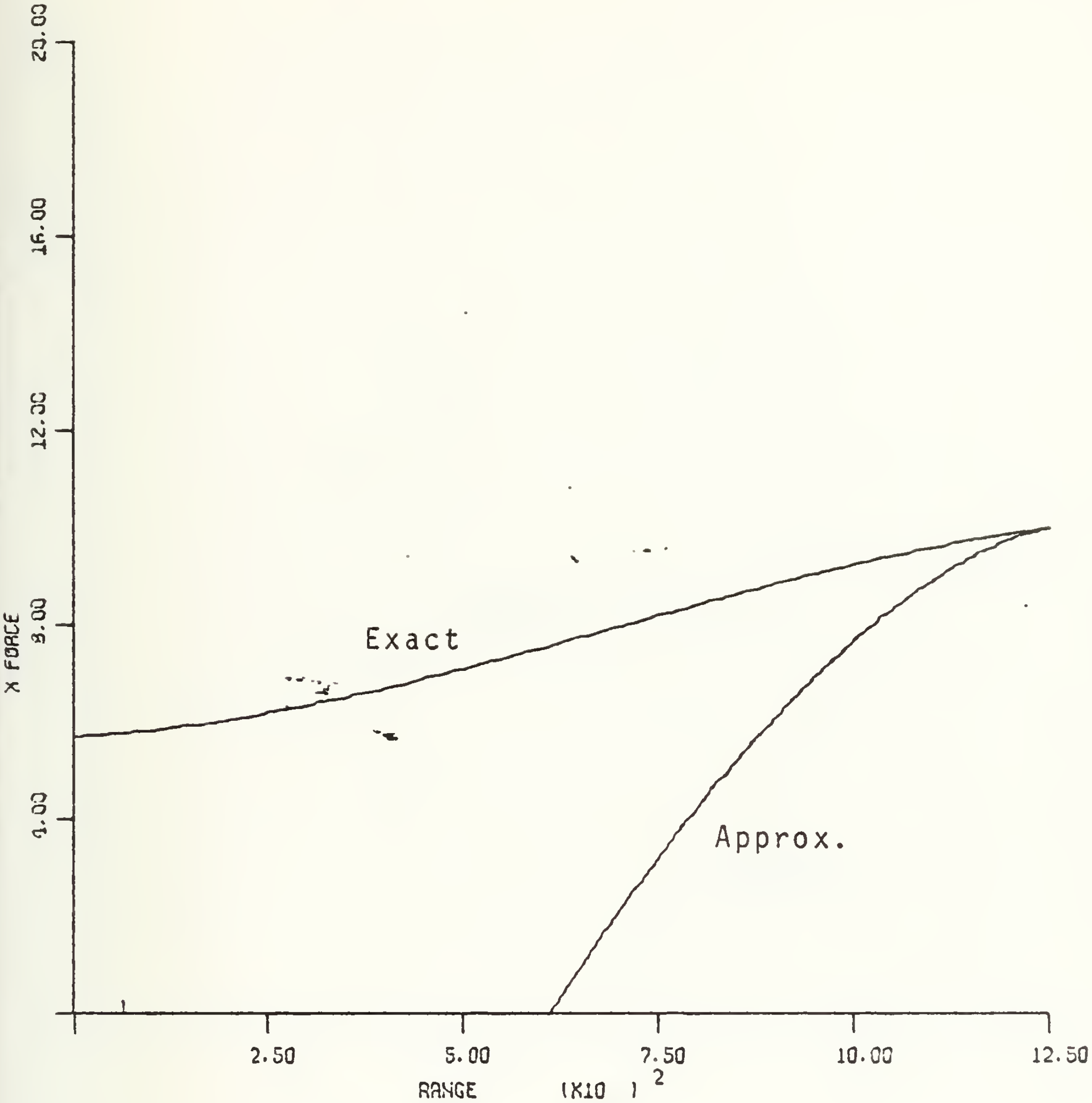

Figure 13. Linear coefficients with R_B = 2000m.

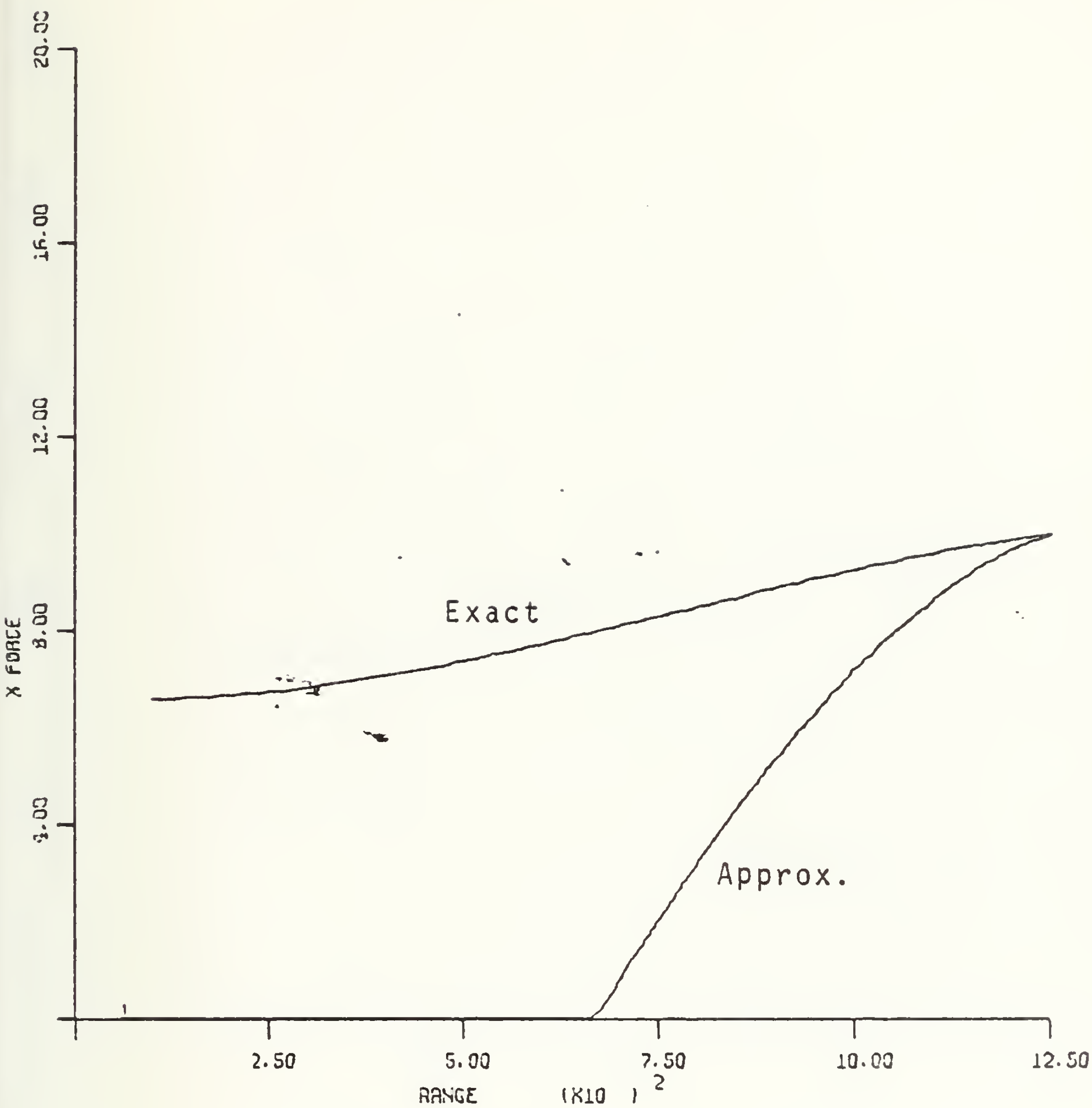

Figure 14. Linear coefficients with R_B = 2500m.

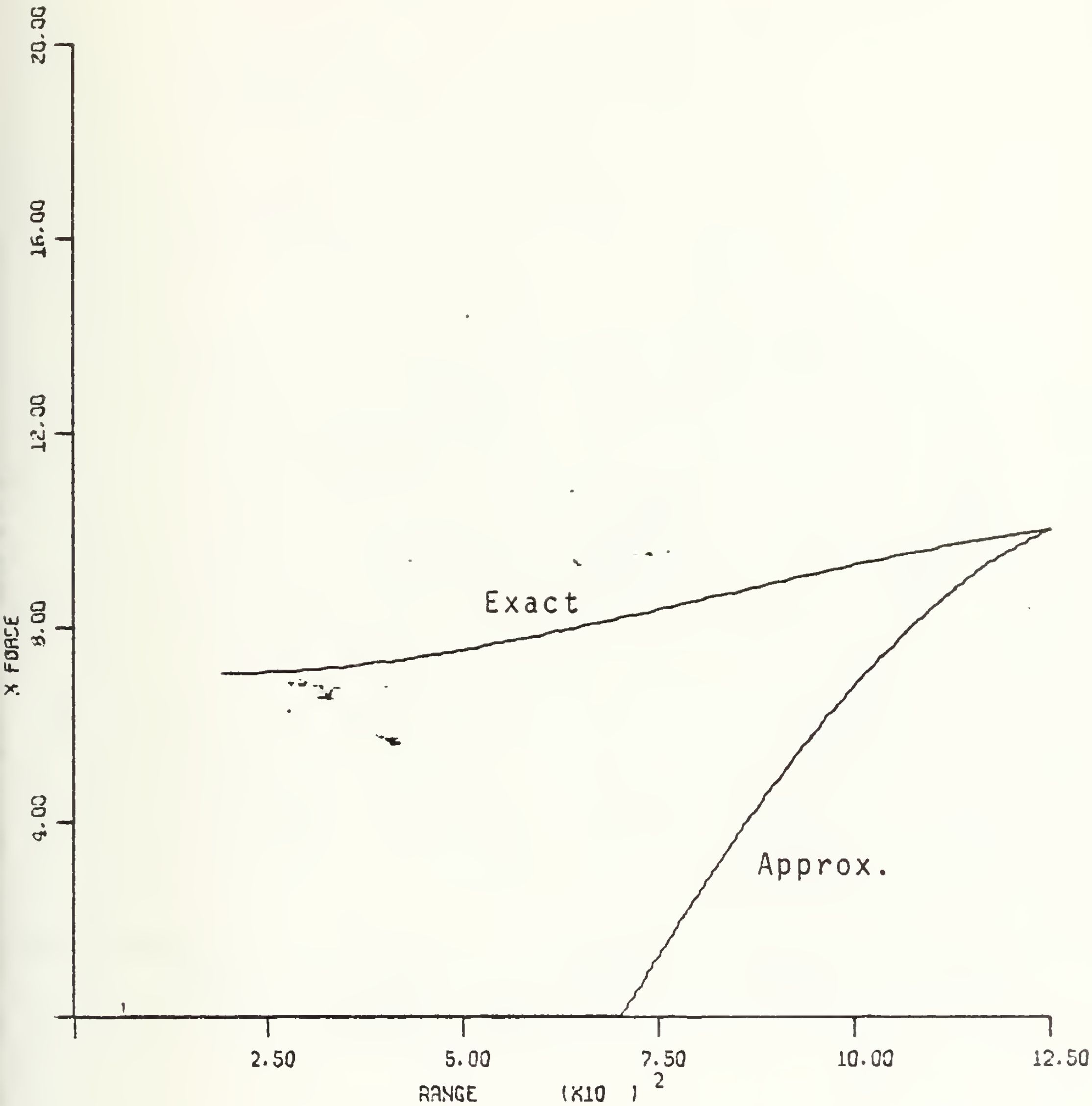

Figure 15. Linear coefficients with R_B = 3000m.

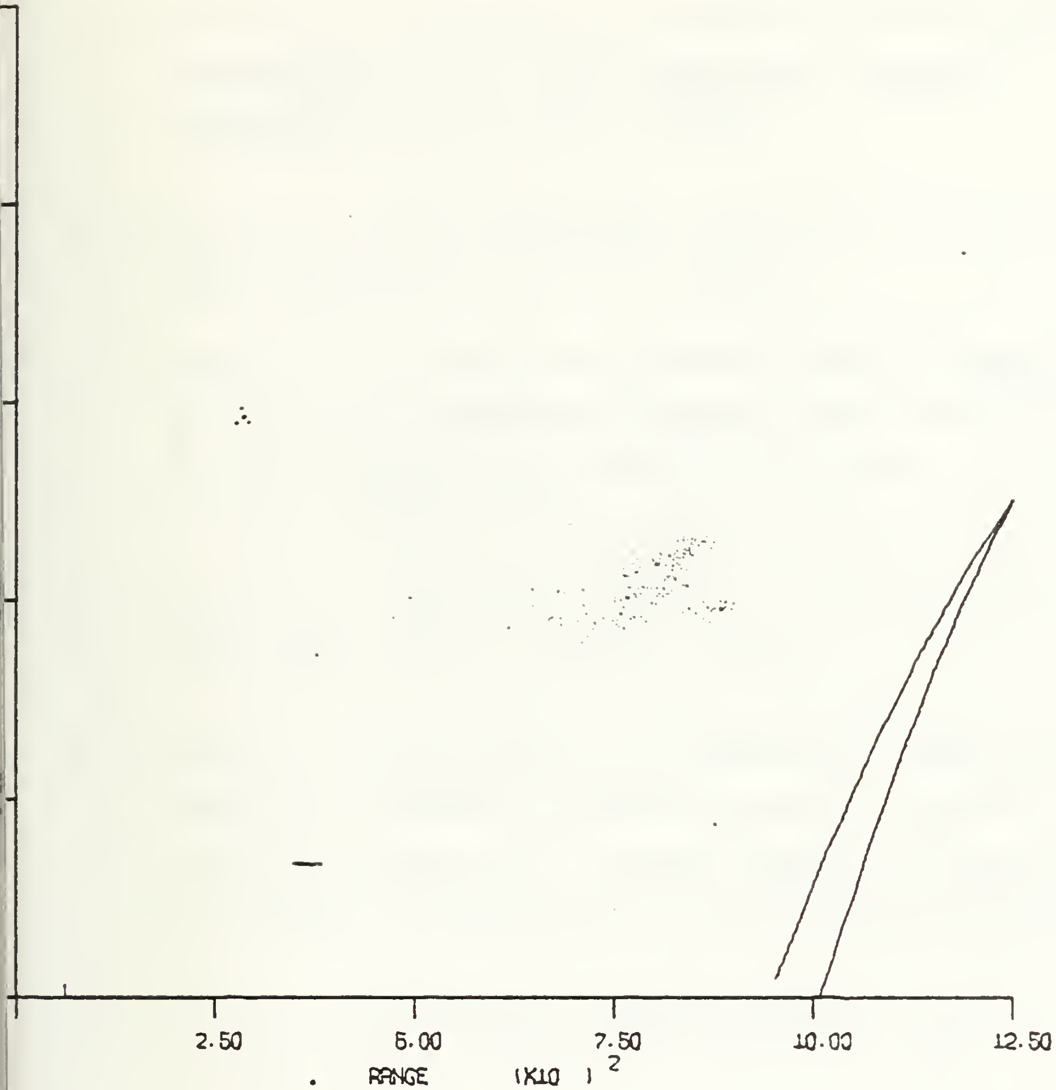

Figure 16. Linear coefficients with R_B=2500m., a_0=b_0=.6

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